Polylogarithms Of Order Three

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In this white paper we will examine the polylogarithm in the following form...

$$Li_{-s}(z)=\sum_{k=1}^{\infty}k^s z^k \text{ ...where... } s\in\{0,1,2,3,4,...\} \text{ ...and... } |z|<1 \tag{1}$$

When the parameter s (order) in Equation (1) above is equal to three then the equation for a polylogarithm of order three is...

$$Li_{-3}(z)=\sum_{k=1}^{\infty}k^3 z^k \text{ ...where... } |z|<1 \tag{2}$$

Our Hypothetical Problem

Given that the parameter $z=0.80$ and the parameter $s=1$ then answer the following questions...

1. What is the value of the polylogarithm over the interval $k=1$ to infinity?

2. What is the value of the polylogarithm over the interval $k=1$ to 4?

Building the Equations

Using Equation (2) above and Appendix Equation (13) below the equation for the value of a polylogarithm of order three over the interval $k=1$ to $k=\infty$ is...

$$Li_{-3}(z)=\sum_{k=1}^{\infty}k^3\,z^k=z\frac{\delta Li_{-2}(z)}{\delta z}=z\,\frac{z^2+4\,z+1}{(1-z)^4}=\frac{z\,(1+4\,z+z^2)}{(1-z)^4} \tag{3}$$

The equation for the value of a polylogarithm of order three over the interval $k=1$ to $k=n$ is...

$$\sum_{k=1}^{n}k^3 z^k=\sum_{k=1}^{\infty}k^3 z^k-\sum_{k=n+1}^{\infty}k^3 z^k \tag{4}$$

Note that we can rewrite the third term in Equation (4) above as...

$$\begin{aligned}\sum_{k=n+1}^{\infty}k^3 z^k&=z^n\sum_{k=1}^{\infty}(k+n)^3 z^k\\&=z^n\sum_{k=1}^{\infty}(k^3+3\,n\,k^2+3\,n^2k+n^3)\,z^k\\&=z^n\sum_{k=1}^{\infty}k^3 z^k+3\,n\,z^n\sum_{k=1}^{\infty}k^2 z^k+3\,n^2 z^n\sum_{k=1}^{\infty}k\,z^k+n^3 z^n\sum_{k=1}^{\infty}z^k\end{aligned} \tag{5}$$

Using Equation (3) above and Appendix Equations (11), (12) and (13) below we can rewrite Equation (5) above as...

$$\begin{aligned}\sum_{k=n+1}^{\infty}k^3 z^k&=z^n\frac{z\,(1+4\,z+z^2)}{(1-z)^4}+3\,n\,z^n\frac{z\,(1+z)}{(1-z)^3}+3\,n^2 z^n\frac{z}{(1-z)^2}+n^3 z^n\frac{z}{1-z}\\&=\frac{z^{n+1}\,(1+4\,z+z^2)}{(1-z)^4}+\frac{3\,n\,z^{n+1}(1+z)}{(1-z)^3}+\frac{3\,n^2 z^{n+1}}{(1-z)^2}+\frac{n^3 z^{n+1}}{1-z}\end{aligned} \tag{6}$$

Using Equations (3) and (6) above we can rewrite Equation (4) above as...

$$\sum_{k=1}^{n} k^3 z^k = \frac{z\,(1+4\,z+z^2)}{(1-z)^4} - \frac{z^{n+1}\,(1+4\,z+z^2)}{(1-z)^4} - \frac{3\,n\,z^{n+1}(1+z)}{(1-z)^3} - \frac{3\,n^2 z^{n+1}}{(1-z)^2} + \frac{n^3 z^{n+1}}{1-z}$$

$$= \frac{(z-z^{n+1})\,(1+4\,z+z^2)}{(1-z)^4} - \frac{3\,n\,z^{n+1}(1+z)}{(1-z)^3} - \frac{3\,n^2 z^{n+1}}{(1-z)^2} - \frac{n^3 z^{n+1}}{1-z} \tag{7}$$

The Answers To Our Hypothetical Problem

1. What is the value of the polylogarithm over the interval $k=1$ to infinity?

Using Equation (3) above the answer to the question is...

$$\sum_{k=1}^{\infty} k^3\,0.80^k = \frac{0.80\times(1+4\times0.80+0.80^2)}{(1-0.80)^4} = 2,420 \tag{8}$$

2. What is the value of the polylogarithm over the interval $k=1$ to 4?

Using Equation (7) above the answer to the question is...

$$\sum_{k=1}^{n} k^3 z^k = \frac{(0.80-0.80^5)\,(1+4\times0.80+0.80^2)}{(1-0.80)^4} - \frac{3\times4\times0.80^5(1+0.80)}{(1-0.80)^3} - \frac{3\times4^2\times0.80^5}{(1-0.80)^2} - \frac{4^3\times0.80^5}{1-0.80}$$

$$= 45.96 \tag{9}$$

References

[1] Gary Schurman, *Polylogarithm Of Order Zero,* May, 2019

[2] Gary Schurman, *Polylogarithm Of Order One,* May, 2019

[3] Gary Schurman, *Polylogarithm Of Order Two,* May, 2019

Appendix

A. The equation for the base polylogarithm is...

$$Li_1 z = \sum_{k=1}^{\infty} k^{-1} z^k = -ln(1-z)\ \text{...where...}\ \frac{\delta Li_1(z)}{\delta z} = \frac{1}{1-z} \tag{10}$$

B. The equation for a polylogarithm of order zero is... [1]

$$Li_0 z = \sum_{k=1}^{\infty} k^0 z^k = \frac{z}{1-z}\ \text{...where...}\ \frac{\delta Li_0(z)}{\delta z} = \frac{1}{(1-z)^2} \tag{11}$$

C. The equation for a polylogarithm of order one is... [2]

$$Li_{-1} z = \sum_{k=1}^{\infty} k^1 z^k = \frac{z}{(1-z)^2}\ \text{...where...}\ \frac{\delta Li_{-1}(z)}{\delta z} = \frac{1+z}{(1-z)^3} \tag{12}$$

D. The equation for a polylogarithm of order two is... [3]

$$Li_{-2} z = \sum_{k=1}^{\infty} k^2 z^k = \frac{z\,(1+z)}{(1-z)^3}\ \text{...where...}\ \frac{\delta Li_{-2}(z)}{\delta z} = \frac{z^2+4\,z+1}{(1-z)^4} \tag{13}$$